Announcements

1) HW 3 up, due next week

2) Completed last integral from 2/14 and put it in the notes

More on Partial Fractions

We figured out

$$\int \frac{1}{(1+x)(1-x)}\,dx$$

by supposing there are numbers

A and B with

$$\frac{1}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x}$$

What about arbitrary rational functions?

If $p(x)$ and $q(x)$ are polynomials, a rational function is

$$\boxed{f(x) = \frac{p(x)}{q(x)}}$$

The degree of a polynomial p, denoted by $\deg(p)$, is the highest power of x appearing in p.

Case 1: $\deg(p) \geq \deg(q)$

Long divide q into p.

You'll be left with a polynomial plus another rational function $g(x) = \dfrac{p_1(x)}{q_1(x)}$

where $\deg(q_1) > \deg(p_1)$.

Integrate the polynomial, go to case 2.

Example of polynomial division

$$p(x) = 7x^5 + 9x^4 - 2x^3 + x + 1$$

$$q(x) = x^2 - 16x + 2$$

multiply

$$\begin{array}{r}
7x^3 + 121x^2 + 1920x + 30478 \\
\hline
x^2 - 16x + 2 \,\big)\, 7x^5 + 9x^4 - 2x^3 + 0x^2 + x + 1 \\
-(7x^5 - 112x^4 + 14x^3) \\
\hline
121x^4 - 16x^3 + 0x^2 + x + 1 \\
-(121x^4 - 1936x^3 + 242x^2) \\
\hline
1920x^3 - 242x^2 + x + 1 \\
-(1920x^3 - 30720x^2 + 3840x) \\
\hline
30478x^2 - 3839x + 1 \\
-(30478x^2 - 487648x + 60956) \\
\hline
483809x - 60955
\end{array}$$

This shows

$$\frac{p(x)}{q(x)} = \frac{7x^5+9x^4-2x^3+x+1}{x^2-16x+2}$$

$$= \overbrace{7x^3+121x^2+1920x+30478}^{\text{polynomial}}$$

$$+ \underbrace{\frac{483809x-60955}{x^2-16x+2}}_{\text{rational function, degree of numerator smaller}}$$

Case 2: $\deg(p) < \deg(q)$

Subcase a) q factors into distinct linear terms

$$q(x) = (a_1x+b_1)\cdot(a_2x+b_2)\cdot \ldots \cdot(a_nx+b_n)$$

where no terms are multiples of any other terms.

There are numbers $A_1, A_2, \ldots, A_n$ with

$$\boxed{\frac{p(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}}$$

Know this!

Example 1: $\int \frac{4x^3+1}{x^2-3x+2}\,dx$ 2 ways

First long-divide! (degree of numerator $\geq$ degree of denominator)

$$
\begin{array}{r}
4x + 12 \\
x^2-3x+2 \,\overline{\big)\, 4x^3 + 0x^2 + 0x + 1} \\
-(4x^3 - 12x^2 + 8x) \\
\hline
12x^2 - 8x + 1 \\
-(12x^2 - 36x + 24) \\
\hline
28x - 23
\end{array}
$$

Then $\displaystyle\int \frac{4x^3+1}{x^2-3x+2}\,dx$

$$= \int \left(4x+12+\frac{28x-23}{x^2-3x+2}\right)dx$$

$$= \int (4x+12)\,dx + \int \frac{28x-23}{x^2-3x+2}\,dx$$

$$= 2x^2+12x + \int \frac{28x-23}{x^2-3x+2}\,dx$$

There are numbers A and B with

$$\frac{28x-23}{x^2-3x+2} = \frac{28x-23}{(x-2)(x-1)}$$

$$= \frac{A}{x-2} + \frac{B}{x-1}$$

Multiply both sides by $(x-2)(x-1)$.

We get

$$28x-23 = A(x-1)+B(x-2)$$

<u>Method 1</u> plug in numbers for x.

Choose the roots of $q(x)$

$$x=1, \ x=2$$

$$28x - 23 = A(x-1) + B(x-2)$$

plug in $x = 1$

$$28 - 23 = B(1-2)$$

so $\boxed{B = -5}$

plug in $x = 2$

$$56 - 23 = A(2-1)$$

so $\boxed{A = 33}$

Method 2: Equate coefficients

$$28x-23 = A(x-1)+B(x-2)$$
$$= Ax - A + Bx - 2B$$
$$= (A+B)x - A - 2B$$

$$A+B = 28$$
$$+(-A-2B = -23) \quad \text{(add to eachother)}$$
$$-B = 5, \quad \boxed{B=-5}.$$

Substitute back

$$A + (-5) = 28, \quad \boxed{A = 33}$$

Back to the integral:

$$\int \frac{4x^2+1}{x^2-3x+2}\,dx$$

$$= 2x^2+12x+\int \frac{28x-23}{x^2-3x+2}\,dx$$

partial fractions

$$= 2x^2+12x+\int\left(\frac{33}{x-2}-\frac{5}{x-1}\right)dx$$

$$= \boxed{2x^2+12x+33\ln|x-2|-5\ln|x-1|+C}$$

Subcase b) $q(x)$ factors into linear terms with repeated factors

$$q(x) = (a_1 x + b_1)^{k_1} \cdot (a_2 x + b_2)^{k_2} \cdot \ldots \cdot (a_n x + b_n)^{k_n}$$

There are constants

$$A_{1,1}, A_{1,2}, \ldots, A_{1,k_1},$$

$$A_{2,1}, A_{2,2}, \ldots, A_{2,k_2},$$

$$\vdots$$

$$A_{n,1}, A_{n,2}, \ldots, A_{n,k_n}$$

with

$$\frac{p(x)}{q(x)} =$$

$$\frac{A_{1,1}}{a_1x+b_1} + \frac{A_{1,2}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1,k_1}}{(a_1x+b_1)^{k_1}} +$$

$$\frac{A_{2,1}}{a_2x+b_2} + \frac{A_{2,2}}{(a_2x+b_2)^2} + \cdots + \frac{A_{2,k_2}}{(a_2x+b_2)^{k_2}} +$$

$$\vdots$$

$$+$$

$$\frac{A_{n,1}}{(a_nx+b_n)} + \frac{A_{n,2}}{(a_nx+b_n)^2} + \cdots + \frac{A_{n,k_n}}{(a_nx+b_n)^{k_n}}$$

Example 2: $\displaystyle\int_4^6 \frac{x^2+2}{(x-3)^2(x+5)^2}\,dx$

There are numbers A, B, C, and D with

$$\frac{x^2+2}{(x-3)^2(x+5)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+5} + \frac{D}{(x+5)^2}$$

multiply both sides by $(x-3)^2(x+5)^2$

$$x^2+2 = A(x-3)(x+5)^2 + B(x+5)^2 + C(x-3)^2(x+5) + D(x-3)^2$$

$$x^2+2 = A(x-3)(x+5)^2 + B(x+5)^2 + C(x-3)^2(x+5)$$
$$+ D(x-3)^2$$

Plug in $x=3$

$$3^2+2 = B(3+5)^2 \quad \text{so}$$

$$11 = B\cdot 64, \quad \boxed{B = \frac{11}{64}}$$

Plug in $x=-5$

$$(-5)^2+2 = D(-5-3)^2$$

$$27 = D\cdot 64, \quad \boxed{D = \frac{27}{64}}$$

Plug in B and D

$$x^2+2 = A(x-3)(x+5)^2 + \frac{11}{64}(x+5)^2 + C(x-3)^2(x+5) + \frac{27}{64}(x-3)^2$$

Plug in 2 other numbers

$x=0$

$$2 = -75A + \frac{275}{64} + 45C + \frac{243}{64}$$

$x=1$

$$3 = -72A + \frac{396}{64} + 24C + \frac{108}{64}$$

$$2 - \frac{518}{64} = -75A + 45C$$

$$3 - \frac{504}{64} = -72A + 24C$$

$$2 - \frac{518}{64} = -75A + 45C$$

$$3 - \frac{504}{64} = -72A + 24C$$

Solve for A in 1st equation

$$-\frac{390}{64} = -75A + 45C$$

$$-75A = -\frac{390}{64} - 45C$$

$$A = \frac{390}{4800} + \frac{45C}{75}$$

$$= \frac{13}{160} + \frac{3C}{5}$$ plus into 2nd equation

$$3 - \frac{504}{64} = -72A + 24C$$

$$A = \frac{13}{160} + \frac{3C}{5}$$

$$-\frac{312}{64} = -72\left(\frac{13}{160} + \frac{3C}{5}\right) + 24C$$

$$= -\frac{117}{20} - \frac{216C}{5} + \frac{120C}{5}$$

$$= -\frac{117}{20} - \frac{94C}{5}$$

$$-\frac{312}{64} + \frac{117}{20} = -\frac{94C}{5}$$

$$C = -\frac{5}{94}\left(-\frac{312}{64} + 117\right)$$

Remember that, after you find the numbers,

$$\int_4^6 \frac{x^2+2}{(x-3)^2(x+5)^2}\,dx$$

$$= \int_4^6 \frac{A}{x-3}\,dx + \int_4^6 \frac{B}{(x-3)^2}\,dx + \int_4^6 \frac{C}{x+5}\,dx + \int_4^6 \frac{D}{(x+5)^2}\,dx$$

$$= \left(A\ln|x-3| - \frac{B}{x-3} + C\ln|x+5| - \frac{D}{x+5} \right)\Bigg|_4^6$$

plug in numbers - -

Subcase c) $q(x)$ cannot be factored into linear terms

There is a result that says any polynomial can be "factored" into linear and quadratic terms which can't be factored further (irreducible quadratics)

This is the worst case and can be done via partial fractions - but we won't do it!

You only need to know how to handle all linear terms!

Improper Integration

Section 7.8

Consider this as a "primer" for infinite series

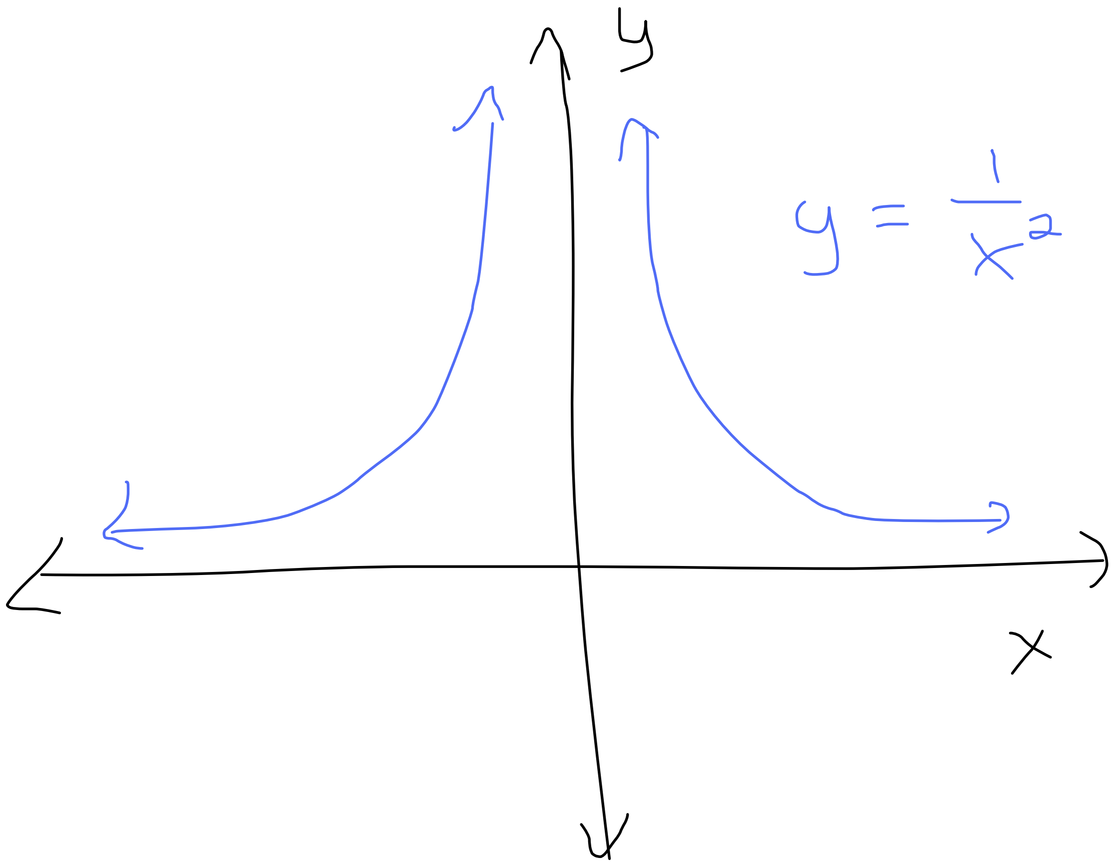
Consider $\displaystyle\int_{-2}^{1} \frac{1}{x^2}\,dx$

$$= \int_{-2}^{1} x^{-2}\,dx$$

$$= -x^{-1} \Big|_{-2}^{1}$$

$$= -1 - \frac{1}{2} = -\frac{3}{2}$$, right?

graph of $f(x) = \frac{1}{x^2}$



Has an asymptote at $x = 0$, which is in the interval from -2 to 1!

Fundamental Theorem of Calculus
does not apply!

Fundamental Theorem is for
continuous functions!

Can we figure out this
integral?

Definition. Suppose f is continuous on $[a, b]$ except maybe at $x = b$

If $a \le t < b$, then $\int_a^t f(x)\,dx$ exists.

Define

$$\boxed{\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx}$$

provided the limit exists!

Example 3: $\displaystyle\int_{-4}^{0} \frac{1}{x^{2/3}}\,dx$

Choose t, $-4 \leq t < 0$.

$$\int_{-4}^{t} x^{-2/3}\,dx = 3x^{1/3}\Big|_{-4}^{t}$$

$$= 3t^{1/3} - 3(-4)^{1/3}$$

$$= 3t^{1/3} + 3(4^{1/3})$$

Then

$$\int_{-4}^{0} \frac{1}{x^{2/3}}\,dx = \lim_{t\to 0^-} \int_{-4}^{t} x^{-2/3}\,dx$$

$$= \lim_{t\to 0^-} \left(3t^{1/3} + 3(4^{1/3}) \right)$$

$$= 3\cdot 0^{1/3} + 3(4^{1/3})$$

$$= \boxed{3(4^{1/3})}$$